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Bifurcation from a Stationary Solution to a 2D Torus for Tearing Unstable Plasmas

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Abstract—We give a mathematical justification, using results of [1,2], of a bifurcation from a stationary solution to a quasiperiodic solution evolving on a 2D torus, for the problem of the double tearing instability of a plasma described by MHD equations. A numerical example of such a transition is then given. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let Ω be a torus in $\mathbb{R}^3$ (see Figure 1), with boundary $\partial\Omega$, containing a plasma described by the one-fluid incompressible magnetohydrodynamics equations,

$$\rho_0\left(\frac{\partial V}{\partial t}+V\cdot\nabla V\right)+\nabla p-\frac{1}{\mu_0}\operatorname{curl}B\times B-\nu\Delta V=0,\qquad \operatorname{div}V=0,$$
$$\frac{\partial B}{\partial t}-\operatorname{curl}(V\times B)+\operatorname{curl}\left(\frac{\eta}{\mu_0}\operatorname{curl}B\right)=0,\qquad \operatorname{div}B=0,$$

where ν is the viscosity, η the resistivity, μ_0 the permeability of the free space, p is the scalar pressure, V is the velocity and B the magnetic field, ρ_0 is the (constant) mass density.

The boundary conditions will concern $V\cdot n$, $B\cdot n$, $\operatorname{curl}B\times n$, and $\operatorname{curl}V\times n$, where n is the outward normal vector at any point of the boundary.

The same problem is also studied if Ω is a cylinder (see Figure 2), with periodic boundary conditions in the direction of the axis of the cylinder.

If Ω is a torus, we note by a and R_0 two characteristic lengths in the poloidal and toroidal directions (for instance the small and large radii of an axisymmetric torus with circular cross-section), if Ω is a cylinder, a is the radius of the cylinder cross-section, and R_0 is defined by $L/(2\pi)$, L being the length of the cylinder. In both cases, we note $\epsilon=a/R_0$.

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The preceding MHD equations will be numerically solved using the 3D code DEMA, we shall see in Section 3, an example of a bifurcation from a stationary solution to a 2D torus, in cylindrical geometry and for a reduced system of MHD equations. A mathematical justification of such a bifurcation is given in Section 2, using a center manifold theorem as in [1,2] the use of center manifolds for tearing instabilities can also be found in [3–5].

Let B_0, p_0 be characteristic values of the magnetic field and of the pressure. We normalize lengths with respect to a, velocity with respect to ϵv_a, where v_a is the Alfven velocity ($v_a = B_0/\sqrt{\rho_0\mu_0}$), time with respect to $(1/\epsilon)\tau_a$, where τ_a is the Alfven time scale ($\tau_a = a/v_a$), B with respect to B_0, p with respect to p_0, viscosity ν with respect to $a^2 v_a \rho_0/R_0$, η with respect to $a^2 v_a \mu_0/R_0$. Let $\beta = 2p_0\mu_0/B_0^2$, we obtain then the normalized equations (as in the DEMA code, see [6])

$$\frac{\partial V}{\partial t} + V\cdot\nabla V + \epsilon^{-2}\left(\frac{\beta}{2\nabla p} - \operatorname{curl} B \times Br\right) - \nu\Delta V = 0, \tag{1.1}$$

$$\operatorname{div} V = 0, \tag{1.2}$$

$$\frac{\partial B}{\partial t} - \operatorname{curl}(V\times B) + \operatorname{curl}(\eta \operatorname{curl} B) = 0, \tag{1.3}$$

$$\operatorname{div} B = 0, \tag{1.4}$$

In these equations, ν is a constant and η is a function, $\eta(r) = \eta_0\hat{\eta}(r)$ with $\eta_0 = \eta(0)$, the coordinate r being defined as in Figures 1 and 2. The ratio $\mathcal{P}_R = \nu/\eta_0$ is the Prandtl number.

2. MATHEMATICAL ANALYSIS OF BIFURCATION

2.1. Application of a Center Manifold Theorem

We note by $\mu = (\eta_0, \nu, \epsilon, \beta, \dots,)$ the set of real parameters, assuming that $\mathcal{P}_R$ is a given constant. Let V^μ_{st}, p^μ_{st}, and B^μ_{st} be some stationary solution of the system such that $V^\mu_{\mathrm{st}}\cdot n = 0$ and $B^\mu_{\mathrm{st}}\cdot n = 0$ on $\partial\Omega$. We set $V = V^\mu_{\mathrm{st}} + U$, $B = B^\mu_{st} + H$, $p = p^\mu_{\mathrm{st}} + 2/(\epsilon^{-2}\beta)\eta_0\tilde{p}$ and we do a time rescale with scaling factor η_0^{-1}; the equations then become

$$\frac{\partial U}{\partial t} = -\nabla\tilde{p} + \mathcal{P}_R\Delta U + \tilde{B}^1_\mu(U,H) + \tilde{N}^1_\mu(U,H), \tag{2.1}$$

$$\operatorname{div} U = 0, \tag{2.2}$$

$$\frac{\partial H}{\partial t} = -\operatorname{curl}(\hat{\eta}\operatorname{curl} H) + B^2_\mu(U,H) + N^2_\mu(U,H), \tag{2.3}$$

$$\operatorname{div} H = 0, \tag{2.4}$$

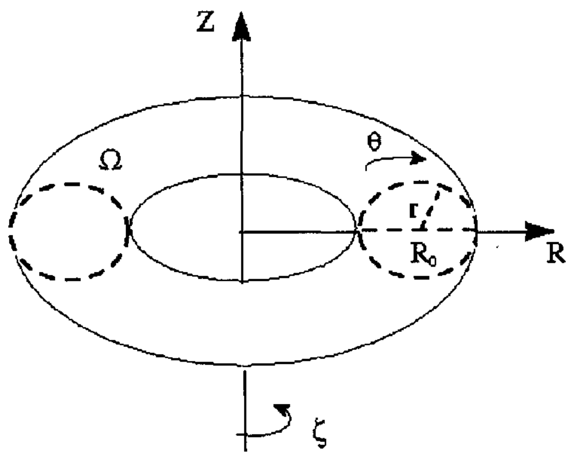

Figure 1.

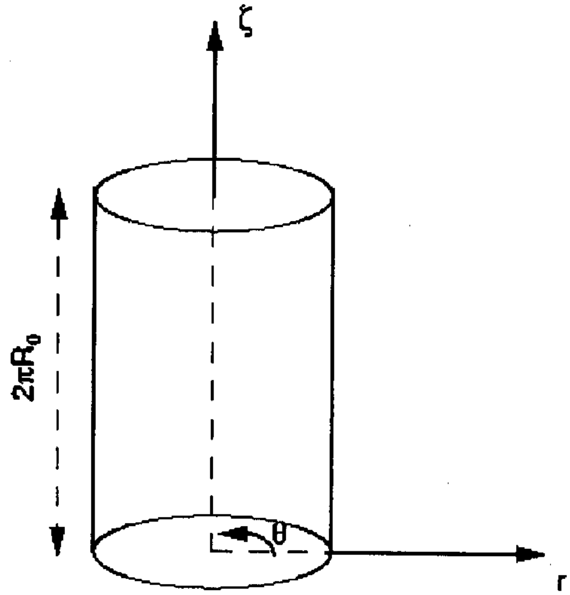

Figure 2.

with boundary conditions $U \cdot n = H \cdot n = 0$, $\operatorname{curl} U \times n = \operatorname{curl} H \times n = 0$ on $\partial\Omega$, and where

$$\begin{aligned}
\tilde{B}^1_\mu(U,H) &= \eta_0^{-1}\epsilon^{-2}\left(\operatorname{curl} B^\mu_{\mathrm{st}} \times H + \operatorname{curl} H \times B^\mu_{\mathrm{st}}\right) - \eta_0^{-1}\left(U \cdot \nabla V^\mu_{\mathrm{st}} + V^\mu_{\mathrm{st}} \cdot \nabla U\right),\\
\tilde{N}^1_\mu(U,H) &= -\eta_0^{-1} U \cdot \nabla U + \eta_0^{-1}\epsilon^{-2} \operatorname{curl} H \times H,\\
B^2_\mu(U,H) &= \eta_0^{-1} \operatorname{curl}\left(U \times B^\mu_{\mathrm{st}} + V^\mu_{\mathrm{st}} \times H\right),\\
N^2_\mu(U,H) &= \eta_0^{-1} \operatorname{curl}(U \times H).
\end{aligned}$$

Let us define the following functional space:

$$\begin{aligned}
Z^1 &= \left\{U \in \left(L^2(\Omega)\right)^3,\ \nabla \cdot U = 0,\ U \cdot n = 0, \text{ on } \partial\Omega\right\},\\
Z^2 &= \left\{H \in Z^1, P_{H_c}(H) = 0\right\}, \qquad Z = Z^1 \times Z^2,
\end{aligned}$$

where $H_c = \{U \in \left(L^2(\Omega)\right)^3,\ \operatorname{curl} U = 0,\ \nabla \cdot U = 0,\ U \cdot n = 0 \text{ on } \partial\Omega\}$ generated by the gradient of a multivalued function (for instance the toroidal angle if Ω is an axisymmetric torus, see [7]). $P_{H_c}(U)$ is the orthogonal projection of U, in $(L^2(\Omega))^3$, on the space H_c; note that we have the following decomposition into orthogonal subspaces (see [7]): $\forall U \in (L^2(\Omega))^3$, there exists unique $U_1 \in Z^2$, $h \in H_c$ $(U_1 + h \in Z^1)$, and $\phi \in H^1(\Omega)$ such that

$$U = U_1 + h + \nabla\phi. \tag{2.5}$$

We also define

$$\begin{aligned}
X^1 &= \left\{U \in \left(H^2(\Omega)\right)^3,\ \nabla \cdot U = 0,\ U \cdot n = 0,\ \operatorname{curl} U \times n = 0, \text{ on } \partial\Omega\right\},\\
X^2 &= \left\{H \in X^1,\ P_{H_c}(H) = 0\right\}, \qquad X = X^1 \times X^2, \quad Y = Z \cap \left(H^1(\Omega)\right)^6.
\end{aligned}$$

Let π_0 be the orthogonal projection from $(L^2(\Omega))^3$ into Z^1, from (2.5), we see that $(I_d - \pi_0)(L^2(\Omega))^3 = \{\nabla\phi, \phi \in H^1(\Omega)\}$. We also see that $Y^1 = \pi_0(H^1(\Omega))^3$. Note that, if $U \in X^1$, $\operatorname{curl}(\operatorname{curl} U)$ is orthogonal to $\nabla(H^1(\Omega))$; then $(I_d - \pi_0)\operatorname{curl}(\operatorname{curl} U) = 0$.

Then, eliminating the pressure $\tilde{p}$ projecting with π_0, the equation of motion becomes: $\frac{\partial U}{\partial t} = A_\mu^1(U,H) + N_\mu^1(U,H)$ where

$$A_\mu^1(U,H) = -\mathcal{P}_R \operatorname{curl}(\operatorname{curl} U) + \pi_0 \tilde{B}_\mu^1(U,H), \qquad N_\mu^1(U,H) = \pi_0 \tilde{N}_\mu^1(U,H).$$

The equation of H is then

$$\frac{\partial H}{\partial t} = A_\mu^2(U,H) + N_\mu^2(U,H), \qquad \text{where } A_\mu^2(U,H) = -\operatorname{curl}(\hat{\eta} \operatorname{curl} H) + B_\mu^2(U,H).$$

Then, for the unknown $u = (U,H)$, we have the following equation, with notations analogous to these of [1, relation (6), paragraph 5]:

$$\frac{du}{dt} = A_\mu(u) + N_\mu(u), \tag{2.6}$$

with $A_\mu = (A_\mu^1, A_\mu^2)$, $N_\mu(u) = (N_\mu^1(U,H), N_\mu^2(U,H))$, and $A_\mu(u) = Tu + B_\mu(u)$ where $Tu = (T^1U, T^2H)$, and $T^1U = -\mathcal{P}_R \operatorname{curl}(\operatorname{curl} U) - U$, $T^2H = -\operatorname{curl}(\hat{\eta}\operatorname{curl} H)$, and $B_\mu^1(u) = \pi_0 \tilde{B}_\mu^1(U,H) + U$. The terms U and $-U$ in the expression of A_μ^1 are introduced to avoid technical difficulties studying the operator T, due to the space H_c. We assume that

$$\hat{\eta} \in W^{1,\infty}(\Omega), \quad 0 < \alpha < \hat{\eta}(x) < \beta < \infty, \qquad \forall x \in \Omega. \tag{2.7}$$

Looking to the principal part of the operator A_μ, the operator T, we see that $T \in \mathcal{L}(X,Z)$. Let $g \in Z$, solving the equation $Tu = g$ for $u \in X$, we can see (as in [8, Chapter 2, paragraph 2, Proposition 2.1]) that there exists one and only one solution u, and $T^{-1} \in \mathcal{L}(Z,X)$. Moreover, noting $(.,.)$ the scalar product in $(L^2(\Omega))^6$, we see that $(Tu,v) = (u,Tv)$, $\forall u,v \in X$, and $(Tu,u) = -\int_\Omega |U|^2 - \int_\Omega \mathcal{P}_R|\operatorname{curl} U|^2 - \int_\Omega \hat{\eta}|\operatorname{curl} H|^2 \le 0$, $\forall u \in X$. Then, as in [1], $Y^1 = \pi_0((H^1(\Omega))^3)$, $N_\mu \in \mathcal{C}^\infty(X,Y)$, $B_\mu \in \mathcal{L}(X,Y)$, assuming that V_{st}^μ and B_{st}^μ are regular enough.

With the preceding properties for N_μ, B_μ, and T, we can, as in [1], apply the center manifold theorem to equation (2.6), for parameter values in the neighbourhood of critical values μ_0 of μ such that the spectrum of A_{μ_0} contains some eigenvalues on the imaginary axis, all the other ones being on the left-hand side of the complex plane.

2.2. Bifurcation Results

We assume now that $\mu = \eta_0^{-1}$, all the other parameters being fixed.

We make the following hypotheses, suggested by numerical results presented in Section 3.

There are only four simple eigenvalues $(\pm i\omega_0)$ and $(\pm i\omega_1)$ of A_{μ_0} on the imaginary axis, the remaining of the spectrum being strictly on the left-hand side of the complex plane. This is exactly the situation studied in [2] (Hypothesis H1—for Navier-Stokes equations), for a problem defined by equation (2.6) (where μ is noted λ, A_μ is $-L_\lambda$, and N_μ is M in the relation (2) of [2]), assuming the properties proved in (2.1) for the operators, and a suitable smooth dependence on the parameter (analyticity), the new dependence of N_μ on μ in our problem would give the same analysis.

We also assume Hypotheses H2 and H3 of [2], implying that two pairs of complex conjugate eigenvalues of A_μ, $\xi_0(\mu), \xi_1(\mu), \overline{\xi}_0(\mu), \overline{\xi}_1(\mu)$, cross the imaginary axis from the left to the right when μ is increasing in a neighbourhood of μ_0, with

$$\xi_0(\mu) = i\omega_0 + (\mu - \mu_0)\xi_0^{(1)} + O(\mu-\mu_0)^2, \qquad \xi_1(\mu) = i\omega_1 + (\mu-\mu_0)\xi_1^{(1)} + O(\mu-\mu_0)^2$$

with $\operatorname{Re}(\xi_0^{(1)}) > 0, \operatorname{Re}(\xi_1^{(1)}) > 0$, and $\omega_1 \ne p\omega_0$ for $p = 2$ or 3 $(\omega_1 > \omega_0 > 0)$.

We can then apply the results of [2] concerning the bifurcation of new solutions.

First, there exists two one-sided bifurcating periodic solutions (limit cycles), denoted by $\mathcal{U}_0$ and $\mathcal{U}_1$, for μ near μ_0.

Then, applying a center manifold theorem for maps, [2] obtains an invariant manifold for a family of maps and also for the dynamical system (2.6). We can also apply directly the results of [1] to the dynamical system (2.6), as in Section 2.1, to obtain the existence of the invariant manifold.

We turn now to the bifurcation results. Normal form analysis is then the same as in [2] (following [9]), and we recall its results.

Adding some new technical hypotheses on the imaginary eigenvalues $i\omega_0$ and $i\omega_1$ ([2, Hypothesis 2]), [2] proves the existence of an invariant torus for the dynamical system. Concerning stability, a new technical assumption proves that for $\mu > \mu_0$, there exists a stable 2D torus and two unstable limit cycles $\mathcal{U}_0$ and $\mathcal{U}_1$.

3. NUMERICAL RESULTS

They are obtained in cylindrical geometry (see Figure 2), with coordinates r, θ, ζ. A static equilibrium solution of (1.1)–(1.4) is given,

$$V_{\mathrm{eq}} = 0, \qquad B_{\mathrm{eq}} = \left(0, \frac{d\psi_{\mathrm{eq}}}{dr}, \mathcal{I}_0\right),$$

where $\mathcal{I}_0$ is a constant and ψ_{eq} is the equilibrium magnetic flux function, depending on r. Rather than to prescribe ψ_{eq}, we choose a hollow current density profile in the ζ direction, given analytically by $J^{\zeta}_{\mathrm{eq}}(r) = J_0(1-r^2)^3(1+9r^2)$, J_0 being a constant. The quantity ψ_{eq} is then an even function of r such that $\psi_{\mathrm{eq}}(1) = 0$ and $\frac{d^2\psi_{\mathrm{eq}}}{dr^2} = J^{\zeta}_{\mathrm{eq}}$ ($J_{\mathrm{eq}} = \operatorname{curl} B_{\mathrm{eq}}$). Defining as usual the safety factor $q(r)$ by $q(r) = rB^{\zeta}_{\mathrm{eq}}/(L/(2\pi)B^{\theta}_{\mathrm{eq}})$, and choosing the constants J_0, $\mathcal{I}_0$ such that $q(0) = 3.8$, $q(1) = 5.46$, we then obtain a nonmonotonic q profile, with two values r_1 and r_2 such that $q(r_1) = q(r_2) = 3$.

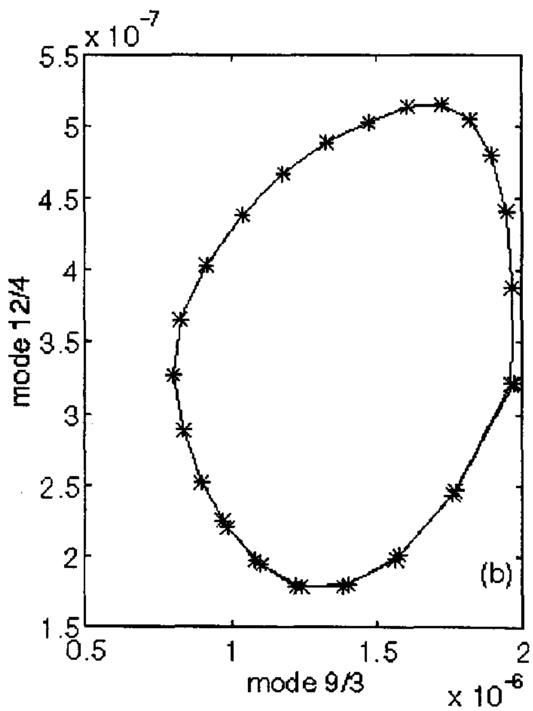

Figure 3.

To satisfy equation (1.3) with the preceding equilibrium quantities, we need to choose η as a function of r such that:

$$\eta(r)\, J^{\zeta}_{\text{eq}}(r) = \text{ constant } = \eta_0 J^{\zeta}_{\text{eq}}(0). \tag{3.1}$$

Then $\mu = \eta_0^{-1}$ will be the bifurcation parameter, increasing from low to high values. We refer to [10] for more details (where we use the Lundquist number $S = (R_0/a)\mu$ instead of μ as bifurcation parameter).

Linear analysis of the stability of such an equilibrium solution physically shows (see [11]) that, for large values of μ, the equilibrium is unstable and that a new solution appears, leading to a change in the topology of the magnetic field in the neighbourhood of the two surfaces r_1 and r_2 defined by $q(r_1) = q(r_2) = 3$ (called "double tearing" instability).

We have done some numerical simulations using the evolution code DEMA (in cylindrical geometry), solving a reduced set of nonlinear MHD equations obtained from equations (1.1)–(1.4) doing some physical assumptions. We need to choose initial conditions and the values of the parameters; to follow the preceding stability analysis we need to choose η as given in (3.1), but to avoid infinite values of η at $r = 1$ we use an approximation of the quantity given by (3.1), satisfying then also assumption (2.7) that we did in the mathematical analysis of bifurcation.

We then numerically compute solutions, and we obtain, for μ greater than some critical value μ_c, a branch of nonlinear stable stationary solutions bifurcating from the equilibrium branch at $\mu = \mu_c$, this is the branch called $(V^{\mu}_{\text{st}}, B^{\mu}_{\text{st}})$ in Section 2. Increasing μ and computing solutions with the evolution code DEMA, we found either the stable stationary solution $(V^{\mu}_{\text{st}}, B^{\mu}_{\text{st}})$, or for higher values of μ, a quasiperiodic solution (with two independent frequencies) evolving on a 2D torus, as can be shown for instance looking to a Poincaré map (see Figure 3) representing successive intersections of a trajectory with some hyperplane in the phase space (see [10] for a complete description of the results). We did not observe limit cycles, suggesting that we have a transition from a stationary branch to a 2D torus, which could be explained by the mathematical analysis of Section 2, assuming some hypothesis on the linearized equations in the neighbourhood of the solution $(V^{\mu}_{\text{st}}, B^{\mu}_{\text{st}})$.

4. CONCLUSION

Numerical results using the evolution MHD code DEMA have shown the existence of a transition from a stationary solution to a quasiperiodic solution evolving on a 2D torus for the double tearing instability. Introducing a convenient variational formulation of the problem, we can apply a center manifold theorem of [1] and the results of [2] to give a mathematical justification of such a transition in terms of bifurcation theory. As indicated in [10], the quasiperiodic solution could explain some reconnection phenomena physically observed.

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